

Poincaré series and monodromy of the simple and unimodal boundary singularities

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Abstract

A boundary singularity is a singularity of a function on a manifold with boundary. The simple and unimodal boundary singularities were classified by V. I. Arnold and V. I. Matov. The McKay correspondence can be generalized to the simple boundary singularities. We consider the monodromy of the simple, parabolic, and exceptional unimodal boundary singularities. We show that the characteristic polynomial of the monodromy is related to the Poincaré series of the coordinate algebra of the ambient singularity.

Introduction

There is a well known classification of hypersurface singularities due to V. I. Arnold. The first classes are the simple and the unimodal singularities. The simple surface singularities in \mathbb{C}^3 are precisely the Kleinian singularities which are quotients of \mathbb{C}^2 by finite subgroups of $SU(2)$. These are weighted homogeneous singularities. Also the parabolic and the 14 exceptional unimodal singularities can be defined by weighted homogeneous equations. To such a singularity, there can be associated two invariants of different nature. One is the Poincaré series of the (graded) coordinate algebra of the singularity. The other one is the characteristic polynomial of the monodromy operator of the singularity. It turned out that there is a close relation between these two invariants. More precisely, it was shown that the Poincaré series of the simple (except A_{2n}) and the 14 exceptional unimodal singularities is the quotient of the characteristic polynomials of two Coxeter elements which are related to the monodromy operators of the singularities. For the simple singularities this was derived from the McKay correspondence.

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Some of these singularities admit a symmetry. This leads to the boundary singularities considered by Arnold in [A1]. Arnold and V. I. Matov classified the simple and unimodal boundary singularities. Here we investigate the relation between the characteristic polynomial of the monodromy and the Poincaré series of the ambient hypersurface singularity for such a singularity. The simple boundary singularities arise from simple hypersurface singularities and there is a generalization of the McKay correspondence for these cases. R. Stekolshchik has extended our theorem for the Kleinian singularities to this generalized McKay correspondence. Here we give an interpretation of his result in terms of singularity theory.

For 7 of the 12 exceptional unimodal boundary singularities we show that there is a direct relation between the Poincaré series of the ambient singularity and the characteristic polynomial of the monodromy.

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1 Simple and unimodal hypersurface singularities

Let $X = \{f(x, y, z) = 0\} \subset \mathbb{C}^3$ be the zero set of a weighted homogeneous function f with weights q_1, q_2, q_3 and degree d . The coordinate algebra $A = \mathbb{C}[x, y, z]/(f)$ has a natural grading $A = \bigoplus_{m=0}^{\infty} A_m$ where A_m consists of the polynomials which are weighted homogeneous of degree m . The series $p_X(t) = \sum_{m=0}^{\infty} \dim A_m \cdot t^m$ is called the Poincaré series of X . One has

$$p_X(t) = \frac{(1 - t^d)}{(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})}.$$

Assume that 0 is an isolated singularity of X . Let B_ε be a ball in \mathbb{C}^3 around the origin of a sufficiently small radius $\varepsilon > 0$, let $\lambda \in \mathbb{C}$ be sufficiently small with $|\lambda| > 0$, and let $X_\lambda := f^{-1}(\lambda) \cap B_\varepsilon$ be the Milnor fibre of the function f . Let $\phi_X(t)$ be the characteristic polynomial of the monodromy operator $c : H_2(X_\lambda) \rightarrow H_2(X_\lambda)$.

If ϕ is a rational function of the form

$$\phi(t) = \prod_{k|d} (1 - t^k)^{\alpha_k} \text{ with } \alpha_k \in \mathbb{Z},$$

then the *Saito dual* ϕ^* of ϕ is defined as

$$\phi^*(t) = \prod_{k|d} (1 - t^{d/k})^{-\alpha_k}.$$

The simple and unimodal singularities of hypersurfaces in \mathbb{C}^3 are classified by Arnold (see [AGV1]) and they fall into the following classes:

- Simple singularities $X = A_\mu, D_\mu, E_6, E_7, E_8$.
- Parabolic singularities $X = \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.
- Hyperbolic singularities $X = T_{p,q,r}$.
- 14 exceptional unimodal singularities.

Except the hyperbolic singularities, all singularities are, for a certain value of the modulus in the unimodal case, given by weighted homogeneous equations. To an exceptional unimodal hypersurface singularity $(X, 0)$ in \mathbb{C}^3 one can associate two triples of numbers. A minimal good resolution of the singularity has an exceptional divisor consisting of a rational curve of self-intersection -1 intersected by three rational curves of self-intersection $-b_1$, $-b_2$, and $-b_3$ respectively [D]. The numbers b_1, b_2, b_3 are called the *Dolgachev numbers* $\text{Dol}(X)$ of the singularity $(X, 0)$. The Coxeter-Dynkin diagram with respect to a certain distinguished basis of vanishing cycles of $H_2(X_\lambda)$ has the shape of Figure 1. The numbers p_1, p_2, p_3 are called the *Gabrielov numbers* $\text{Gab}(X)$ of the singularity [G]. Here each vertex represents a sphere of self-intersection number -2 , two vertices connected by a single solid edge have intersection number 1, and two vertices connected by a double broken line have intersection number -2 . The intersection number is 0 if the corresponding vertices are not connected. *Arnold's strange duality* is the observation that there is an involution $X \mapsto X^*$ on the set of the 14 exceptional unimodal singularities such that

$$\text{Dol}(X) = \text{Gab}(X^*), \quad \text{Gab}(X) = \text{Dol}(X^*).$$

Note that a singularity of type $T_{p,q,r}$ has a Coxeter-Dynkin diagram of the form of Figure 1 with $p_1 = p$, $p_2 = q$, $p_3 = r$ and the vertex with label μ omitted. The characteristic polynomial of the monodromy of a singularity of type $T_{p,q,r}$ is equal to

$$\phi_{T_{p,q,r}}(t) = (1-t)^{-1}(1-t^p)(1-t^q)(1-t^r).$$

Theorem 1 *The Poincaré series of the singularity $(X, 0)$ is a quotient*

$$p_X = \frac{\phi}{\psi}$$

of two polynomials in the variable t in the following cases:

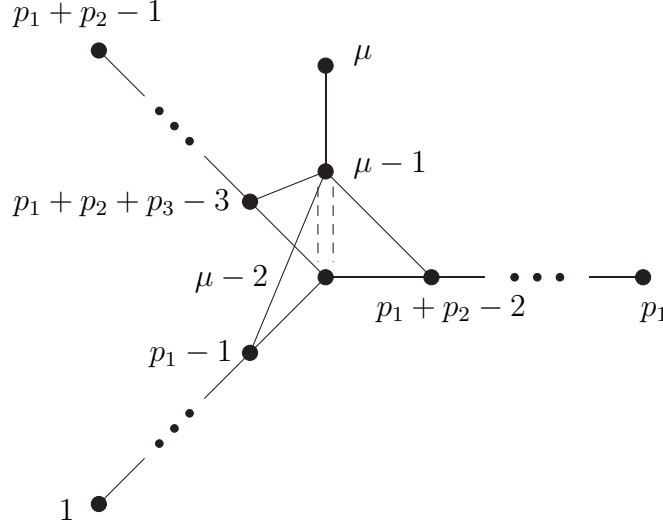


Figure 1: Coxeter-Dynkin diagram of an exceptional unimodal singularity

- (i) $(X, 0)$ is simple. If $(X, 0)$ is not of type A_{2n} then $\phi = \phi_X = \phi_X^*$ is the characteristic polynomial of the Coxeter element of the corresponding root system and ψ is the characteristic polynomial of the Coxeter element of the corresponding affine root system.
- (ii) $(X, 0)$ is parabolic. In this case, $\phi = \phi_X^*$ and $\psi = 1$.
- (iii) $(X, 0)$ is an exceptional unimodal singularity. In this case, $\phi = \phi_{X^*} = \phi_X^*$ and $\psi = \phi_{T_{p,q,r}}$ where p, q, r are the Dolgachev numbers of the singularity $(X, 0)$.

Remark 1 Note that in Theorem 1 (i) and (iii), the Poincaré series is the quotient of the characteristic polynomials of the Coxeter elements corresponding to two Coxeter-Dynkin diagrams which differ only by one vertex. In Case (i) the denominator diagram has one vertex more and in Case (iii) one vertex less than the numerator diagram.

Proof. (i) is [E2, Theorem 2], (ii) follows from [E2, Example 1], and (iii) follows from [E2, Theorem 1] and [E3, Proposition 1 and Remark 1]. Note that (iii) also follows from results of H. Lenzing and J. de la Peña [LdlP]. See also the forthcoming paper [EP]. \square

2 McKay correspondence

Theorem 1(i) can be derived from the McKay correspondence, see [E2]. Let $G \subset SU(2)$ be a finite subgroup. Then $(X, 0) = (\mathbb{C}^2/G, 0)$ is one of the simple surface singularities. If G is a cyclic group of order $\mu + 1$ then $(X, 0)$ is of type A_μ . In this case, the weights and the degree of $(X, 0)$ are not unique. We assume that a singularity $(X, 0)$ of type A_μ is given by the standard equation $f(x, y, z) = x^{\mu+1} + y^2 + z^2 = 0$. If G is not cyclic of odd order (i.e. $(X, 0)$ is not of type A_{2n}) then we have

$$p_X(t^2) = P_G(t) = \sum_{m=0}^{\infty} \dim S^m(\mathbb{C}^2)^G \cdot t^m,$$

where $P_G(t)$ is the Poincaré series of the algebra of invariants of the group G . Let ρ_0, \dots, ρ_n be the inequivalent irreducible representations of G where ρ_0 is the trivial representation. Denote by $\rho : G \rightarrow SU(2)$ the given natural representation. Define an $(n+1) \times (n+1)$ -matrix $B = (b_{ij})$ by decomposing the tensor products

$$\rho_j \otimes \rho = \oplus_{i=0}^n b_{ij} \rho_i$$

into irreducible components. Define the matrix C by $B = 2I - C$. McKay [McK] observed that the matrix C is the affine Cartan matrix corresponding to a simply laced root system R . In [E2] the following theorem is proved.

Theorem 2 *Let G be not cyclic of odd order. Then the Poincaré series P_G satisfies*

$$P_G(t) = \frac{\phi(t^2)}{\psi(t^2)}$$

where ϕ is the characteristic polynomial of the Coxeter element and ψ is the characteristic polynomial of the affine Coxeter element of the corresponding root system R .

Slodowy [Sl1, Sl2] has found a generalization of the McKay correspondence to include the non simply-laced Coxeter-Dynkin diagrams. We recall Slodowy's construction.

Let $H \subset SU(2)$ be a finite subgroup and let $G \triangleleft H$ be a normal subgroup of H . If $\rho : H \rightarrow GL(V)$ is a representation of H , denote by ρ^\downarrow its restriction to the subgroup G . Let ρ_0, \dots, ρ_m be the inequivalent irreducible representations of H where ρ_0 is the trivial representation. Let $\rho : G \rightarrow SU(2)$ be the given natural representation. Let $\rho_0^\downarrow, \dots, \rho_n^\downarrow$ be the different inequivalent restrictions to G of the representations ρ_0, \dots, ρ_m . Define an $(n+1) \times (n+1)$ -matrix $B = (b_{ij})$ by decomposing the tensor products

$$\rho_j^\downarrow \otimes \rho = \oplus_{i=0}^n b_{ij} \rho_i^\downarrow$$

G	$X = \mathbb{C}^2/G$	H	R	p_X	ψ_R	ϕ_R
\mathcal{C}_{2n}	A_{2n-1}	$\tilde{\mathcal{D}}_n$	B_n	$2n/1 \cdot n \cdot n$	$2 \cdot (n-1)$	$2n/n$
$\tilde{\mathcal{D}}_{n-1}$	D_{n+1}	$\tilde{\mathcal{D}}_{2(n-1)}$	C_n	$2n/2 \cdot (n-1) \cdot n$	$1 \cdot n$	$2n/n$
$\tilde{\mathcal{T}}$	E_6	$\tilde{\mathcal{O}}$	F_4	$12/3 \cdot 4 \cdot 6$	$2 \cdot 3$	$2 \cdot 12/4 \cdot 6$
$\tilde{\mathcal{D}}_2$	D_4	$\tilde{\mathcal{T}}$	G_2	$6/2 \cdot 2 \cdot 3$	$1 \cdot 2$	$1 \cdot 6/2 \cdot 3$

Table 1: Generalized McKay correspondence

into irreducible components. Define the matrix C by $B = 2I - C$. Then in the cases listed in Table 1, C is the affine Cartan matrix corresponding to the dual R^\vee of the root system R in the table.

The following generalization of Theorem 2 is due to R. Stekolshchik [St].

Theorem 3 (Stekolshchik) *Let G , H , and R be as in Table 1. Then the Poincaré series P_G satisfies*

$$P_G(t) = \frac{\phi_{R^\vee}(t^2)}{\psi_{R^\vee}(t^2)}$$

where ϕ_{R^\vee} is the characteristic polynomial of the Coxeter element and ψ_{R^\vee} is the characteristic polynomial of the affine Coxeter element of the corresponding dual root system R^\vee .

In Table 1 we use the following notation: \mathcal{C}_{2n} denotes the cyclic group of order $2n$ ($n \geq 1$), $\tilde{\mathcal{D}}_k$ the binary dihedral group of order $4k$ ($k \geq 2$), $\tilde{\mathcal{T}}$ the binary tetrahedral group, and $\tilde{\mathcal{O}}$ the binary octahedral group. A rational function $\prod_{k|d} (1 - t^k)^{\alpha_k}$ ($\alpha_k \in \mathbb{Z}$) is represented by its *Frame shape* $\prod_{k|d} k^{\alpha_k}$. We have indicated p_X ($P_G(t) = p_X(t^2)$), ψ_R , and ϕ_R . Note that $B_n^\vee = C_n$, $C_n^\vee = B_n$, $F_4^\vee = F_4$, and $G_2^\vee = G_2$.

We want to interpret Theorem 3 in terms of singularities. The group H/G acts in a natural way on the quotient $X = \mathbb{C}^2/G$. The pair $(X, H/G)$ is a singularity with symmetry. In particular, if $H/G \cong \mathbb{Z}_2$ then the pair $(X, H/G)$ is a simple boundary singularity in the sense of Arnold [A1]. We shall consider such singularities in the next section.

3 Boundary singularities

Let \bar{f} be a singularity of a function on a manifold with boundary. This means that \bar{f} is the germ of a holomorphic function $\bar{f} : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ such that

the function \bar{f} has an isolated critical point at zero both in the space \mathbb{C}^{n+1} and on the subspace \mathbb{C}^n given by the equation $x_0 = 0$, where x_0, x_1, \dots, x_n are the coordinates of \mathbb{C}^{n+1} . Then \bar{f} defines a germ of a function $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ invariant under the involution $\sigma : (x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$ by

$$f(x_0, x_1, \dots, x_n) := \bar{f}(x_0^2, x_1, \dots, x_n).$$

Let $(X, 0)$ be the singularity defined by $f = 0$. We call it the *ambient singularity* of the boundary singularity \bar{f} . The involution σ defines a natural \mathbb{Z}_2 -action on $(X, 0)$. From now on we shall assume $n = 2$ and we denote the coordinates of \mathbb{C}^3 by x, y, z .

We recall some facts about Coxeter-Dynkin diagrams and monodromy of boundary singularities from [AGV2]. The middle homology group $H_2(X_\lambda) = H_2(X_\lambda; \mathbb{Z})$ of a Milnor fibre X_λ of $(X, 0)$ splits as an orthogonal direct sum

$$H_2(X_\lambda) = H^+ \oplus H^-,$$

where H^+ and H^- are the subgroups consisting of invariant and antiinvariant cycles respectively with respect to $\sigma_* : H_2(X_\lambda) \rightarrow H_2(X_\lambda)$. There exists a basis $\delta_1^{(1)}, \dots, \delta_{\mu_0}^{(1)}, \delta_1^{(2)}, \dots, \delta_{\mu_0}^{(2)}, \delta'_1, \dots, \delta'_{\mu_1}$ of vanishing cycles of $H_2(X_\lambda)$ such that

$$\sigma_*(\delta_i^{(1)}) = \delta_i^{(2)}, \quad i = 1, \dots, \mu_0; \quad \sigma_*(\delta'_j) = -\delta'_j, \quad j = 1, \dots, \mu_1,$$

and the intersection matrix with respect to this basis has the form

$$\begin{pmatrix} A & 0 & B \\ 0 & A & -B \\ B^t & -B^t & A' \end{pmatrix}$$

where A and A' are the intersection matrices with respect to the elements $\delta_1^{(1)}, \dots, \delta_{\mu_0}^{(1)}$ and $\delta'_1, \dots, \delta'_{\mu_1}$ respectively and B is the matrix $(\langle \delta_i^{(1)}, \delta'_j \rangle)_{i=1, \dots, \mu_0}^{j=1, \dots, \mu_1}$. $\langle \cdot, \cdot \rangle$ denotes the intersection form on $H_2(X_\lambda)$. Set

$$\widehat{\delta}_i := \delta_i^{(1)} - \delta_i^{(2)}, \quad i = 1, \dots, \mu_0.$$

Then the sublattice H^- of $H_2(X_\lambda)$ spanned by $\widehat{\delta}_1, \dots, \widehat{\delta}_{\mu_0}, \delta'_1, \dots, \delta'_{\mu_1}$ can be considered as the Milnor lattice of the boundary singularity \bar{f} . The Coxeter-Dynkin diagram of the boundary singularity \bar{f} corresponding to this basis is obtained by *folding* the Coxeter-Dynkin diagram corresponding to the basis $\delta_1^{(1)}, \dots, \delta_{\mu_0}^{(1)}, \delta_1^{(2)}, \dots, \delta_{\mu_0}^{(2)}, \delta'_1, \dots, \delta'_{\mu_1}$ (see Figure 2). Note that

$$\langle \widehat{\delta}_i, \widehat{\delta}_i \rangle = -4, \quad \langle \widehat{\delta}_i, \widehat{\delta}_j \rangle = 2\langle \delta_i^{(1)}, \delta_j^{(1)} \rangle \text{ for } 1 \leq i, j \leq \mu_0, i \neq j,$$

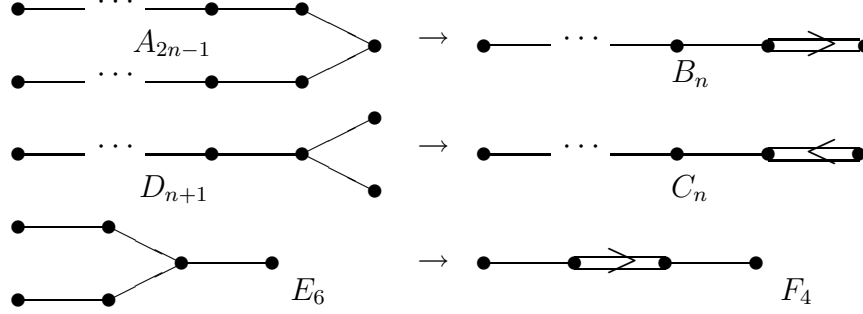


Figure 2: Coxeter-Dynkin diagrams of B_n , C_n , and F_4 obtained by folding

$$\langle \widehat{\delta}_i, \delta'_j \rangle = 2\langle \delta_i^{(1)}, \delta'_j \rangle \text{ for } 1 \leq i \leq \mu_0, 1 \leq j \leq \mu_1.$$

Therefore the reflection $s_{\widehat{\delta}_i}$ corresponding to an element $\widehat{\delta}_i$ ($1 \leq i \leq \mu_0$),

$$s_{\widehat{\delta}_i}(x) = x - \frac{2\langle x, \widehat{\delta}_i \rangle}{\langle \widehat{\delta}_i, \widehat{\delta}_i \rangle} \widehat{\delta}_i \text{ for } x \in H^-,$$

is a well defined operator $s_{\widehat{\delta}_i} : H^- \rightarrow H^-$. The Coxeter element (product of reflections) corresponding to the basis $\widehat{\delta}_1, \dots, \widehat{\delta}_{\mu_0}, \delta'_1, \dots, \delta'_{\mu_1}$ of H^- can be considered as the monodromy of the boundary singularity. Let $\phi_{\overline{f}}$ be its characteristic polynomial. Let ϕ_1 be the characteristic polynomial of the Coxeter element corresponding to $\widehat{\delta}_1, \dots, \widehat{\delta}_{\mu_0}$.

Proposition 1 *One has*

$$\phi_{\overline{f}} = \frac{\phi_X}{\phi_1}.$$

Proof. Write $A = -V^t - V$, $A' = -V'^t - V'$ for upper triangular matrices V, V' with diagonal entries equal to 1. Then the formula of [B, Chap. V, §6, Exercice 3] for the characteristic polynomial of a Coxeter element yields

$$\begin{aligned} \phi_X(t) &= \begin{vmatrix} -V^t - tV & 0 & tB \\ 0 & -V^t - tV & -tB \\ B^t & -B^t & -V'^t - tV' \end{vmatrix} \\ &= \det(-V^t - tV) \begin{vmatrix} -V^t - tV & -2tB \\ -B^t & -V'^t - tV' \end{vmatrix} = \phi_1(t) \phi_{\overline{f}}(t). \end{aligned}$$

□

If \overline{f} is a boundary singularity then there is a dual boundary singularity defined by the Lagrange transform \overline{f}^\vee of \overline{f} . This is defined as follows (see,

e.g., [A2]). Let $\tilde{f}(x, y, z, p) = px + \bar{f}(x, y, z)$ be the function in an additional variable p with the equation of the boundary being $p = 0$. Then \tilde{f} can be written as $\tilde{f} = \bar{f}^\vee(x', y', z') + w^2$ for a function \bar{f}^\vee with an isolated singularity on the boundary $x' = 0$. S. Chmutov and I. Scherback [CS] have proved the following theorem.

Theorem 4 (Chmutov, Scherback) *Lagrange dual boundary singularities have dual Coxeter-Dynkin diagrams (in the sense that the arrows are reverted).*

Corollary 1 *The characteristic polynomials of the monodromy of dual boundary singularities are equal.*

Proof. One can easily see that the reversion of arrows in the Coxeter-Dynkin diagram does not change the characteristic polynomial of the corresponding Coxeter element. \square

4 Simple and unimodal boundary singularities

The simple and unimodal boundary singularities have been classified by Arnold and V. I. Matov. The simple boundary singularities are the singularities B_n, C_n, F_4 [A1] where the notation is derived from a corresponding Coxeter-Dynkin diagram. The singularities B_n are dual to the singularities C_n and F_4 is self-dual.

As a corollary of Theorem 1 and Proposition 1 we obtain the following reformulation of Theorem 3 in the cases B_n, C_n , and F_4 . Let \bar{f} be a simple boundary singularity with ambient singularity $(X, 0)$. Let ψ_X be the characteristic polynomial of the Coxeter element of the affine root system corresponding to $(X, 0)$ and let $\phi_1 = \frac{\phi_X}{\phi_{\bar{f}}}$. Define $\psi_{\bar{f}} := \frac{\psi_X}{\phi_1}$.

Corollary 2 *If \bar{f} is a simple boundary singularity then the Poincaré series of the ambient singularity $(X, 0)$ satisfies*

$$p_X = \frac{\phi_{\bar{f}}}{\psi_{\bar{f}}}.$$

According to Matov [Ma] the weighted homogeneous unimodal boundary singularities fall into two classes. There are 3 parabolic ones: a dual pair $F_{1,0} \leftrightarrow L_6 = D_{4,1}$ and a self-dual one denoted by $K_{4,2}$. Moreover, there are

Name	Ambient	p_X	$\psi_{\bar{f}}$	$\phi_{\bar{f}}$	Dual
B_n	A_{2n-1}	$2n/1 \cdot n \cdot n$	$1 \cdot n$	$2n/n$	C_n
C_n	D_{n+1}	$2n/2 \cdot (n-1) \cdot n$	$2 \cdot (n-1)$	$2n/n$	B_n
F_4	E_6	$12/3 \cdot 4 \cdot 6$	$2 \cdot 3$	$2 \cdot 12/4 \cdot 6$	F_4
$F_{1,0}$	\tilde{E}_8	$6/1 \cdot 2 \cdot 3$		$3 \cdot 3$	$L_6 = D_{4,1}$
$K_{4,2}$	\tilde{E}_7	$4/1 \cdot 1 \cdot 2$		$2 \cdot 4$	$K_{4,2}$
$L_6 = D_{4,1}$	\tilde{E}_6	$3/1 \cdot 1 \cdot 1$		$3 \cdot 3$	$F_{1,0}$
F_8	E_{14}	$24/3 \cdot 8 \cdot 12$	$3 \cdot 4$	$4 \cdot 24/8 \cdot 12$	$E_{6,0}$
F_9	$J_{3,0}$	$18/2 \cdot 6 \cdot 9$	$2 \cdot 6$	$18/9$	$E_{7,0}$
F_{10}	E_{18}	$30/3 \cdot 10 \cdot 15$		$15/5$	$E_{8,0}$
K_8^*	W_{13}	$16/3 \cdot 4 \cdot 8$	$3 \cdot 4$	$16/8$	D_5^1
K_9^*	$W_{1,0}$	$12/2 \cdot 3 \cdot 6$	$2 \cdot 6$	$12/3$	$E_{6,1}$
K_8^{**}	W_{12}	$20/4 \cdot 5 \cdot 10$	$2 \cdot 5$	$2 \cdot 20/4 \cdot 10$	K_8^{**}
$E_{6,0}$	Q_{10}	$24/6 \cdot 8 \cdot 9$		$4 \cdot 24/8 \cdot 12$	F_8
$E_{7,0}$	Q_{11}	$18/4 \cdot 6 \cdot 7$		$18/9$	F_9
$E_{8,0}$	Q_{12}	$15/3 \cdot 5 \cdot 6$	$3 \cdot 6$	$15/5$	F_{10}
D_5^1	S_{11}	$16/4 \cdot 5 \cdot 6$		$16/8$	K_8^*
$E_{6,1}$	U_{12}	$12/3 \cdot 4 \cdot 4$	$4 \cdot 4$	$12/3$	K_9^*
D_4^2	U_{12}	$12/3 \cdot 4 \cdot 4$		$4 \cdot 12/2 \cdot 6$	D_4^2

Table 2: Simple, parabolic, and exceptional unimodal boundary singularities

12 exceptional ones, see Table 2. The corresponding ambient singularities are either unimodal (8 different singularities, one occurring twice) or bimodal (3 cases). There are 5 dual pairs and 2 self-dual singularities. Looking at Table 2 one observes that if the ambient singularities are dual in Arnold's sense then the corresponding boundary singularities are Lagrange dual. The Poincaré series p_X and the characteristic polynomials $\phi_{\bar{f}}$ are also indicated in Table 2.

Theorem 5 *For 7 of the 12 exceptional unimodal boundary singularities, the*

Name	Ambient	Gabrielov	Dolgachev	$\psi_{\bar{f}}$	Arnold's dual	Dual
F_8	E_{14}	2 3 9	3 3 4	$3 \cdot 4$	Q_{10}	$E_{6,0}$
F_9	$J_{3,0}$	2 3 10	2 2 2 3	$2 \cdot 6$		$E_{7,0}$
F_{10}	E_{18}					$E_{8,0}$
K_8^*	W_{13}	2 5 6	3 4 4	$3 \cdot 4$	S_{11}	D_5^1
K_9^*	$W_{1,0}$	2 6 6	2 2 3 3	$2 \cdot 6$		$E_{6,1}$
K_8^{**}	W_{12}	2 5 5	2 5 5	$2 \cdot 5$	W_{12}	K_8^{**}
$E_{6,0}$	Q_{10}	3 3 4	2 3 9		E_{14}	F_8
$E_{7,0}$	Q_{11}	3 3 5	2 4 7		Z_{13}	F_9
$E_{8,0}$	Q_{12}	3 3 6	3 3 6	$3 \cdot 6$	Q_{12}	F_{10}
D_5^1	S_{11}	3 4 4	2 5 6		W_{13}	K_8^*
$E_{6,1}$	U_{12}	4 4 4	4 4 4	$4 \cdot 4$	U_{12}	K_9^*
D_4^2	U_{12}	4 4 4	4 4 4		U_{12}	D_4^2

Table 3: Gabrielov and Dolgachev numbers of the ambient singularities

Poincaré series of the ambient singularity $(X, 0)$ is a quotient

$$p_X = \frac{\phi_{\bar{f}}}{\psi_{\bar{f}}}$$

where $\psi_{\bar{f}}$ is the characteristic polynomial of the Coxeter element of a folded $T_{p,q,q}$ - or $T_{p,p,q}$ -diagram having one vertex less than the Coxeter-Dynkin diagram of \bar{f} .

Proof. For $F_8, K_8^*, K_8^{**}, E_{8,0}$, and $E_{6,1}$ this follows from Theorem 1 by folding the Dolgachev graph of the ambient singularity (cf. Table 3).

For K_9^* the ambient singularity is $W_{1,0}$. It has a Coxeter-Dynkin diagram which is an extension of the $T_{2,6,6}$ -diagram [E1]. The statement for this case can be derived from this fact.

For F_9 the polynomial $\psi_{\bar{f}}$ is the same as the corresponding polynomial of K_9^* . \square

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